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ON THE ASYMPTOTIC STABILITY AND INSTABILITY OF THE ZERO SOLUTION OF A NON-AUTONOMOUS SYSTEM WITH RESPECT TO PART OF THE VARIABLES*

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A non-autonomous set of differential equations is considered which allows of the existence of a set of differential equations limiting it. Theorems of the asymptotic stability and instability of the zero solution of such systems with respect to part of the variables are proved in the presence of Liapunov function with derivatives of constant sign. Sufficient conditions are obtained for the partial asymptotic stability of non-autonomous holonomic mechanical system subjected to the action of dissipative forces with total or partial dissipation. The problem of the asymptotic stability of the equilibrium of a heavy solid with a fixed point in a homogeneous gravitational field of variable intensity, and of stabilizing the axis of symmetry of a symmetric satellite perpendicular to the orbital plane of the latter, whose centre of mass remains at the libration points of the limited circular three-body problem, are considered as examples.

1. Consider the set of differential equations

$$x^{\cdot} = X(t, x) \quad (X(t, 0) \equiv 0) \tag{1.1}$$
$$x = (x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_p)$$
$$(m > 0,\ p \geqslant 0,\ n = m + p)$$

The vector function $X(t, x)$ is defined in the region $R^+ \times \Gamma$ ($R^+ = [0, +\infty[$, where $\Gamma = \{\|y\| < H,\ H > 0,\ \|z\| < +\infty\}$ and $\|y\|$ is some norm R^m, $\|z\|$ — в R^p, $\|x\| = \|y\| + \|z\|$), that satisfies in that region the conditions of z-continuability /1/ and conditions (A) from /2/. The latter ensures the existence and uniqueness of solutions of (1.1), the existence of functions $\varphi(t, x)$ limiting $X(t, x)$, the reciprocal continuity of the solution of the input system (1.1), and of solutions of the limiting systems

$$x^{\cdot} = \varphi(t, x) \tag{1.2}$$

We shall also assume that the non-negative scalar function $W(t, x)$ $(W(t, 0) \equiv 0)$ used below satisfies conditions (A). The limiting function for $W(t, x)$ is denoted by $\omega(t, x)$ /3/.

We shall call (φ, ω) the limiting pair, if $\varphi(t, x)$ and $\omega(t, x)$ are limits for $X(t, x)$ and $W(t, x)$ of one and the same sequence $t_n \to +\infty$.

2. For each limiting pair (φ, ω) we denote the set formed by the non-continuable solutions of system $x^{\cdot} = \varphi(t, x)$ that lies in the whole of its range of definition on the set $\{\omega(t, x) = 0,\ t \in R^+,\ x \in \Gamma\}$ by $M^+((\varphi, \omega))$, and by $M^+(\{(\varphi, \omega)\})$ the union $M^+(\varphi, \omega)$ over all (φ, ω).

Theorem 2.1. Let us assume that: 1) the solution of (1.1) in some neighbourhood Γ_1 of the point $x = 0$ is bounded by z, 2) a y-positive definite function $V(t, x)$, $V(t, x) \geqslant V_1(\|y\|)$ exists whose derivative by virtue of (1.1) is $V^{\cdot}(t, x) \leqslant -W(t, x) \leqslant 0$, and 3) for any limiting pair (φ, ω) the set $M^+(\varphi, \omega) \subset \{x : y = 0\}$. The zero solution of (1.1) is then asymptotically y-stable.

Proof. Condition (2) implies that the zero solution of (1.1) is asymptotically stable /4/. Let $x = x(t, t_0, x_0)$ be a solution of (1.1) from the neighbourhood $\Gamma(t_0)$ of point $x = 0$ so that $\sup (V(t, x)$ for $x \in \Gamma(t) \subset \Gamma_1) \leqslant V_1(H_2)$, when $H_2 < H$. By conditions (1) and (2) of the theorem, it is bounded for all $t \geqslant t_0$. By Theorem 2.2. of /3/ the set of limiting points of the solution $\Omega^+(x(t, t_0, x_0))$ is contained in $M_*^+(\{(\varphi, \omega)\})$. But by condition (3) of the theorem $M_*^+(\{(\varphi, \omega)\}) \subset \{x : y = 0\}$. Hence $\Omega^+(x(t, t_0, x_0)) \subset \{x : y = 0\}$ and $\lim y(t, t_0, x_0) = 0$ as $t \to +\infty$.

Definition. Let us define form some sequence $t_n \to -\infty$ and any $c \geqslant 0$ and $t \geqslant 0$ the limiting set $N(t, c)$, as the set of points $x \in \Gamma$, for which the sequence $x_n \to x$ exists so that $\lim V(t_n + t, x_n) = c$ as $t_n \to +\infty$ and $x_n \to x$.

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Theorem 2.2. Let us assume that: 1) the solutions of (1.1) in some neighbourhood of $x = 0$ are bounded by z, 2) the y-positive definite function $V(t, x)$ whose derivative by virtue of (1.1) is $V^{\cdot}(t, x) \leqslant -W(t, x) \leqslant 0$, and 3) for some sequence $t_n \to +\infty$ the limiting pair (φ_0, ω_0) and the set $N(t, c)$ are such that for any $c_0 > 0$ the set $N(t, c_0) \cap \{\omega_0(t, x) = 0\}$ does not contain solutions of system $x^{\cdot} = \varphi_0(t, x)$. The zero solution of (1.1) is asymptotically y-stable, uniformly with respect to x_0.

Theorem 2.3. Let us assume that: 1) the solutions of (1.1) from some neighbourhood of $x = 0$ are bounded by z, 2) a function $V(t, x)$ exists that, in any arbitrarily small neighbourhood of $x = 0$, takes positive values, is bounded in the region $V(t, x) \geqslant 0$, and whose derivative by virtue of (1.1) is $V^{\cdot}(t, x) \leqslant - W(t, x) \leqslant 0$, and 3) for some sequence $t_n \to +\infty$ the limiting pair (φ_0, ω_0) and the limiting set $N(t, c)$ are such that for any $c_0 > 0$ the set $N(t, c_0) \cap \{\omega_0(t, x) = 0\}$ does not contain solutions of the system $x^{\cdot} = \varphi_0(t, x)$. The zero solution of (1.1) is then y-unstable.

The proof of Theorems 2.2. and 2.3 is a modification of the proof of Theorems 3.1 and 3.2 in /5/. Thus, when proving Theorem 2.2 we have shown that along any bounded solution $x = x(t, t_0, x_0)$ of (1.1), the function $V(t, x(t, t_0, x_0)) \downarrow 0$, which according to /6/ implies the asymptotic y-stability, uniform with respect to x_0.

Suppose that the function $V(t, x)$ satisfies the Lipschitz condition with respect to t and x on every compact $K = [t_0, t_0 + T] \times \Gamma_1$ $(t_0 \geqslant 0, T > 0, \Gamma_1 \subset \Gamma)$. Then a function $\rho(t, x)$ exists which is limiting for $V(t, x)$ in the sense of uniform convergence on each compact K as $t_n \to +\infty$ /3/. We shall call (φ, ρ, ω) the limiting set, if simultaneously $X(t_n + t, x) \to \varphi(t, x)$, $V(t_n + t, x) \to \rho(t, x)$, $W(t_n + t, x) \to \omega(t, x)$ for some sequence $t_n \to +\infty$.

Theorem 2.4. Let us assume that: 1) the solutions of (1.1) from some neighbourhood Γ_1 of the point $x = 0$ are uniformly bound by z, 2) a y-positive definite function $V(t, x)$ exists that satisfies the Lipschitz condition with respect to t and x (and consequently, admits of an infinitely small higher limit), $V_1(\| y \|) \leqslant V(t, x) \leqslant V_2(\| x \|)$, whose derivative by virtue of (1.1) is $V^{\cdot}(t, x) \leqslant -W(t, x) \leqslant 0$, and 3) for any limiting set (φ, ρ, ω) the set $\{\rho(t, x) = c > 0\} \cap \{\omega(t, x) = 0\}$ does not contain solutions of system $x^{\cdot} = \varphi(t, x)$. The zero solution of (1.1) is then asymptotically y-stable.

Proof. The theorem implies the uniform y-stability of the zero solution of (1.1) and, also, that solutions of (1.1) from $\Gamma_0 = V_2^{-1}(V_1(H_1)) \subset \Gamma_1$ $(H_1 < H)$ are bounded. By Theorem 2.2 we obtain that along each solution of (1.1) from Γ_0 the function $V(t, x(t, t_0, x_0)) \downarrow 0$. The theorem will be proved, if we can show that $V(t, x(t, t_0, x_0)) \to 0$ uniformly with respect to $t_0 \in R^+$ and $x_0 \in \Gamma_0$.

For this we first derive the following result. Let $(\varphi_0, \rho_0, \omega_0)$ be an arbitrary limiting set. Then along every solution $x = \psi(t, t_0, x_0)$, $x_0 \in \Gamma_0$ of the system $x^{\cdot} = \varphi_0(t, x)$ the function $\rho_0(t, \psi(t, t_0, x_0)) \downarrow 0$.

Repeating the reasoning of Theorem 3.3 of /3/, we obtain

$$\rho_0(t, \psi(t, t_0, x_0)) - \rho_0(t_0, x_0) \leqslant -\int_0^t \omega_0(\tau, \psi(\tau, t_0, x_0)) \, d\tau \leqslant 0$$

From this it follows that the solution of system $x^{\cdot} = \varphi_0(t, x)$ is y-stable, and its solutions from Γ_0 are bounded.

The system that is limiting for $x^{\cdot} = \varphi_0(t, x)$ is also limiting for (1.1), as are functions limiting for $\rho_0(t, x)$, and $\omega_0(t, x)$ are limiting for $V(t, x)$ and $W(t, x)$. Hence from condition (3) we have that, if $(\varphi', \rho', \omega')$ is the limiting set for $(\varphi_0, \rho_0, \omega_0)$, the set $\{\rho'(t, x) = c_0 > 0\} \cap \{\omega'(t, x) = 0\}$ does not contain solutions of system $x^{\cdot} = \varphi'(t, x)$. From this and Theorem 2.2. it follows that $\rho_0(t, \psi(t, t_0, x_0)) \downarrow 0$.

Let us now assume the contrary: $V(t, x(t, t_0, x_0)) \to 0$ as $t \to +\infty$ non-uniformly with respect to $t_0 \in R^+$ and $x_0 \in \Gamma_0$, i.e. an $\varepsilon_0 > 0$ exists such that for the sequence $T_n \to +\infty$ a sequence (t_n, x_n), $t_n \geqslant 0$, $x_n \in \Gamma_0$ can be found for which $V(t_n + T_n, x(t_n + T_n, t_n, x_n)) \geqslant \varepsilon_0$. Then, evidently, for $t, t_n \leqslant t_n + T_n$ we have

$$V(t, x(t, t_n, x_n)) \geqslant \varepsilon_0 \tag{2.1}$$

Using the compactness of Γ_0, we select the subsequence $\{x_k\}$ so that $x_k \to x_0^* \in \Gamma_0$. The sequence $\{t_k\}$ cannot be bounded, since that would contradict the property $V(t, x(t, t_0, x_0^*)) \downarrow 0$ of continuity of the solutions of (1.1) from the initial conditions and the continuity of $V(t, x)$.

Let $t_k \to +\infty$. We select $\{t_j\} \subset \{t_k\}$ so as to have $X(t_j + t, x) \to \varphi_0(t, x)$, $V(t_j + t, x) \to \rho_0(t, x)$, $W(t_j + t, x) \to \omega_0(t, x)$. The sequence $x_j(t) = x(t_j + t, t_j, x_j)$ of the solution of (1.1) convergest uniformly in $t \in [0, T]$ $(T > 0)$ to the solution $x = \psi(t, 0, x^*)$ of system $x^\cdot = \varphi_0(t, x)$. By virtue of (2.1) $V(t_j + t, x_j(t)) \geqslant \varepsilon_0$ when $t \in [0, T_j]$. Passing to the limit as $t_j \to +\infty$, we obtain that $\rho(t, \psi(t, 0, x^*)) \geqslant \varepsilon_0 > 0$ for all $t \geqslant 0$. But this contradicts the property $\rho(t, \psi(t, 0, x_0^*)) \downarrow 0$ obtained above. Thus $V(t, x(t, t_0, x_0)) \to 0$ uniformly with respect to (t_0, x_0), which proves the theorem.

Theorem 2.5. Suppose that: 1) there exists a y-positive definite function $V(t, x)$, $V(t, x) \geqslant V_1(\|y\|)$ whose derivative by virtue of (1.1) is $V^\cdot(t, x) \leqslant -W(t, x) \leqslant 0$, 2) $V(t, x)$ is such that $N > 0$ exists for any $\delta > 0$ and any sequence $\{x_n\}$ such that $\|y_n\| \geqslant \delta$ as $\|z_n\| \to +\infty$, and $\underline{\lim}\, V(t, x_n) \geqslant N$ uniformly with respect to t, and 3) for any limiting pair (φ, ω) the set $M^+((\varphi, \omega)) \subset \{x : y = 0\}$. The zero solution of (1.1) is then asymptotically y-stable.

Proof. Let $x = x(t, t_0, x_0)$, where $x_0 \in \Gamma(t_0)$ $(\Gamma(t) : \sup(V(t, x)$ when $x \in \Gamma(t)) < \inf(N, V_1(H_1))$, $H_1 < H)$, be the solution of (1.1). Then $V(t, x(t, t_0, x_0)) \leqslant V_0 = V(t_0, x_0)) < N$. If we assume the existence or a sequence $t_n \to +\infty$ for which $\|z(t_n, t_0, x_0)\| \to +\infty$ and $\|y(t_n, t_0, x_0)\| \geqslant \delta_0 > 0$, the inequality $V(t_n, x(t_n, t_0, x_0)) \leqslant V_0 < N$ contradicts condition (2) of the theorem. If, however, $\|z(t_n, t_0, x_0)\|$ is bounded, the set of limiting points $\Omega^+(x(t, t_0, x_0))$ is non-empty. By virtue of condition (3), on the basis of Theorem 2.1 of /3/ we have $\Omega^+(x(t, t_0, x_0)) \subset \{x : y = 0\}$. Hence, even when $\|z(t_n, t_0, x_0)\|$ is bounded, we have $\lim y(t_n, t_0, x_0) = 0$ as $t_n \to +\infty$.

The following theorem can be proved similarly.

Theorem 2.6. Let us assume that: 1) the function $V(t, x)$ exists and, in an arbitrarily small neighbourhood of $x = 0$, takes positive values, and is bounded in the region $V(t, x) > 0$, with a derivative which by virtue of (1.1) is $V^\cdot(t, x) \geqslant W(t, x) \geqslant 0$ and such that uniformly on $t - \overline{\lim}\, V(t, x) \leqslant 0$ as $\|z\| \to +\infty$, and (2) a sequence $t_n \to +\infty$ exists for which the limiting set $N(t, c)$ and the limiting pair (φ_0, ω_0) are such that for any $c_0 > 0$ the set $N(t, c_0) \cap \{\omega_0(t, x) = 0\}$ does not contain solutions of the system $x^\cdot = \varphi_0(t, x)$. The zero solution of (1.1) is then y-unstable.

The theorems considered above generalize and develop the theorems on asymptotic stability and instability with respect to part of the variables with the Lyapunov function having a derivative of constant sign /5—9/.

3. Consider a mechanical system with time dependent constraints, defined by the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q^\cdot}\right) - \frac{\partial L}{\partial q} = Q \tag{3.1}$$
$$q = (q_1, q_2, \ldots, q_n)^T, \quad L = L_2 + L_1 + L_0$$
$$L_2 = {}^1/_2\, (q^\cdot)^T A(t, q)\, q^\cdot, \quad L_1 = B^T(q)\, q^\cdot, \quad L_0 = L_0(t, q)$$
$$(\|q\|^2 = q_1^2 + q_2^2 + \ldots + q_n^2)$$

where $Q(t, q, q^\cdot)$ is the resultant of generalized gyroscopic and dissipative forces, and $Q^T \cdot q^\cdot \leqslant 0$; $\partial L/\partial q \equiv 0$, $Q \equiv 0$ when $q^\cdot = q = 0$ so that the system has a zero position of equilibrium $q^\cdot \equiv q \equiv 0$.

Let us assume that $L_0(t, 0) \equiv 0$, $\partial L/\partial t \geqslant 0$, so that for the derivative of the function $L_2 - L_0$ we have

$$(L_2 - L_0)^\cdot = -\partial L/\partial t + Q^T \cdot q^\cdot \leqslant Q^T \cdot q^\cdot$$

We shall also assume that the quantitites $A(t, q)$, $\partial A/\partial t$, $\partial A/\partial q$, $\partial B/\partial q$, $\partial L_0/\partial q$, Q are bounded and satisfy the Lipschitz conditions by all of their variables. The limiting equations for (3.1) then exist and have the form /3/

$$A_*{}^T q^{\cdot\cdot} + \{(q^\cdot)^T C_* q^\cdot\} + \{D_*{}^T q^\cdot\} + F_* = Q_* \tag{3.2}$$
$$F_*(t, q) = \lim_{t_n \to +\infty} \frac{\partial L_0}{\partial q}(t_n + t, q)$$

For convenience we denote by $\beta_1(t)$ a function such that $\beta_1(t) \geqslant 0$, $\beta_1(t) \geqslant \beta_0 > 0$ when $t \in [t_n, t_n + \nu]$ $(\nu > 0$, $t_n \to +\infty$, $t_{n+1} - t_n \leqslant \rho = \text{const}$, by $\beta_2(t)$ a function such that $\beta_2^\cdot(t) \geqslant 0$, $\beta_2(t) \geqslant \beta_0 > 0$ when $t \in [t_n, t_n + \nu]$ (when $(\nu > 0$, $t_n \to +\infty$ condition $t_{n+1} - t_n \leqslant \rho$ is not satisfied).

Theorem 3.1. Assume that: 1) the function $V = -L_0(t, q)$ is positive definite relative to $q_1, q_2, \ldots, q_m$ $(m < n)$, 2) the motions (3.1) from some neighbourhood of $q^\cdot = q = 0$ are bounded by $q_{m+1}, \ldots, q_n$, 3) there are no equilibrium positions outside the set $\{q_1 = q_2 = \ldots = q_m = 0\}$, and this property is non-degenerate, i.e. for any $\varepsilon > 0$ a $\delta = \delta(\varepsilon) > 0$ exists such that $\|\partial L_0/\partial q\| \geqslant \delta$ when $q_1^2 + q_2^2 + \ldots + q_m^2 \geqslant \varepsilon$, and 4) the dissipative forces are such that $Q^T \cdot q^\cdot \leqslant -\alpha(\|q^\cdot\|)$. The zero equilibrium Position (3.1) is asymptotically stable with respect to $q^{\cdot}, q_1, q_2, \ldots, q_m$.

Theorem 3.2. Assum that: 1) the function $V = -L_0(t, q)$ is positive definite with respect to $q_1, q_2, \ldots, q_m$ $(m < n)$, 2) motions in some neighbourhood of $q^{\cdot} = q = 0$ are uniformly bounded by $q_{m+1}, \ldots, q_n$, 3) there are no equilibrium positions (3.1) on the set $L_0(t, q) < 0$ and this property is non-degenerate, i.e. for any $\varepsilon > 0$ an $\delta = \delta(\varepsilon) > 0$ exists such that $\|\partial L_0/\partial q\| > \delta$ when $L_0(t, q) \leqslant -\varepsilon$, and 4) the dissipative forces are such that $Q^T \cdot q^{\cdot} \leqslant -\beta_1(t)\,\alpha(\|q^{\cdot}\|)$. The zero equilibrium position is then uniformly asymptotically stable with respect to $q^{\cdot}, q_1, q_2, \ldots, q_m$.

If $Q^T \cdot q^{\cdot} \leqslant -\beta_2(t)\,\alpha(\|q^{\cdot}\|)$, then $q^{\cdot} = q = 0$ is uniformly stable with respect to $(q_0, q_0^{\cdot})$ and asymptotically stable with respect to $q^{\cdot}, q_1, q_2, \ldots, q_m$.

The proof of Theorems 3.1 and 3.2 follows from Theorems 2.1, 2.2, and 2.4. Theorem 2.5 enables us to substitute conditions relative to $L_0(t, q)$ for the condition of boundedness of the solutions in Theorem 3.1.

Theorem 3.3. If under conditions (1)–(3) of Theorem 3.1 the conditions (2') is also satisfied for any $\delta > 0$ $\underline{\lim}\,(-L_0(t, q)) = N > 0$ when $q_1^2 + q_2^2 + \ldots + q_m^2 \geqslant \delta > 0$ and $q_{m+1}^2 + q_{m+2}^2 + \ldots + q_n^2 \to +\infty$ uniformly relative to t, then the zero solution of (3.1) is stable with respect to $q^{\cdot}$ and asymptotically stable with respect to $q_1, q_2, \ldots, q_m$. If the right sides of (3.1), solved for $q^{\cdot\cdot}$, are bounded, then $q^{\cdot} = q = 0$ is asymptotically stable with respect to $q^{\cdot}$. It was assumed in the preceding theorems that the quadratic part of the Lagrangian function L_2 is positive definite for all velocities, as is usually the case. Sometimes, however, the properties of the motions of mechanical systems may be conveniently investigated in a system of coordinates in which L_2 degenerates, becoming positive definite not for all coordinate values with respect to all velocities /10/. The question of stability in such cases was considered in detail in /11/.

Let us assume that $L_2(t, q, q^{\cdot})$ is positive definite with respect to $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_k^{\cdot}$ $(k < n)$ when $q_1 = q_2 = \ldots = q_m = 0$, and positive definite relative to all velocities when $q_1^2 + q_2^2 + \ldots + q_m^2 \geqslant \delta > 0$. Then $L_2(t, q, q^{\cdot}) \to +\infty$ when $q_1^2 + q_2^2 + \ldots + q_m^2 \geqslant \delta$, and $(q_{k+1}^{\cdot})^2 + (q_{k+2}^{\cdot})^2 + \ldots + (q_n^{\cdot})^2 \to +\infty$ uniformly with respect to t. Hence on the basis of Theorem 2.5 we have the following theorem.

Theorem 3.4. In the case considered here under conditions (1)-(4) of Theorem 3.1 the zero equilibrium position (3.1) is stable with respect to $q_1^{\cdot}, q_2^{\cdot}, \ldots, q_k^{\cdot}$ and asymptotically stable with respect to $q_1, q_2, \ldots, q_m$.

Remark 3.1. Theorems 3.1-3.4 may be suitably extended to systems with partial dissociation. For instance, Theorems 3.1, 3.3, and 3.4) hold, if instead of conditions 3) and 4) we have conditions 3') $Q^T \cdot q^{\cdot} \leqslant -\alpha\left(((q_i^{\cdot})^2 + (q_{i+1}^{\cdot})^2 + \ldots + (q_j^{\cdot})^2)^{1/2}\right)$ $(1 \leqslant i \leqslant j \leqslant n)$, and 4') the motions of the limiting systems (3.2) along which $q_i^{\cdot} \equiv q_{i+1}^{\cdot} \equiv q_{i+2}^{\cdot} \equiv \ldots \equiv q_j^{\cdot} \equiv 0$ are contained in the set $\{q_1 = q_2 = \ldots = q_m = 0\}$.

Example 3.1. The problem of partial asymptotic stability of the zero equilibrium position of a heavy material point moving on the surface $z = (1 + x^2)\,y^2$ was considered in /12, 7, 9/. Supplementing the data of these papers by using Theorem 3.3 and Remark 3.1 we can conclude that the equilibrium position $x^{\cdot} = y^{\cdot} = x = y = 0$ is stable with respect to $x^{\cdot}, y^{\cdot}$ and y, and asymptotically stable with respect to $y^{\cdot}$ and y under the action of forces Q_x and Q_y such that $Q_x x^{\cdot} + Q_y y^{\cdot} \leqslant -\beta_1(t)\,\alpha(|y^{\cdot}|)$.

Example 3.2. Consider a solid with a single fixed point. The centre of mass of the body lies on one of the principal axes of inertia, the x axis, in a uniform gravitational field of variable intensity $g = g(t) \geqslant g_0 > 0$, and subjected to the moment of the force of resistance $\mathbf{M} = -k(t)\,\omega^{\alpha}\boldsymbol{\omega}_0$ of the medium (ω is the angular velocity of the solid and $\boldsymbol{\omega}_0$ is the corresponding unit vector). The position of the solid in the inertial system of coordinates is defined by Euler's angles θ, φ, ψ. The Lagrangian function

$$\begin{aligned} L &= L_2 + L_0,\quad 2L_2 = A\,\omega_x^2 + B\omega_y^2 + C\omega_z^2 = \\ &\quad A(\psi^{\cdot}\sin\theta\sin\varphi + \theta^{\cdot}\cos\varphi)^2 + \\ &\quad B(\psi^{\cdot}\sin\theta\cos\varphi - \theta^{\cdot}\sin\varphi)^2 + C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta)^2 \\ L_0 &= -mg(t)\,(1 + \sin\theta\sin\varphi) \end{aligned}$$

where A, B, C are the principal central moments of inertia, m is the mass, and $x_0 > 0$ is the coordinate of the centre of mass of the solid. The solid is in the equilibrium position when the x axis is directed vertically downward

$$\theta^{\cdot} = \varphi^{\cdot} = \psi^{\cdot} = 0,\quad \theta = \frac{\pi}{2},\quad \varphi = \frac{3\pi}{2},\quad \psi = 0 \tag{3.4}$$

It follows from Theorem 3.2 that under conditions $g^{\cdot}(t) \leqslant 0,\ k(t) = \beta_1(t)$(i.e. $k(t)$ a function of the type $\beta_1(t)$) is asymptotically stable with respect to $\theta^{\cdot}, \varphi^{\cdot}, \psi^{\cdot}, \theta, \varphi$. In the more general case we take $V = L_3/g(t) + mx_0(1 + \sin\theta\sin\varphi)$ as the Lyapunov function. For small $\omega_x, \omega_y, \omega_z$ its derivative is

$$V^{\cdot} = (-2k(t)g(t)\omega^{\alpha+1} - g^{\cdot}(t)(A\omega_x^2 + B\omega_y^2 + C\omega_z^2))/2g^2(t) \leqslant -\beta_1(t)\omega^{\alpha+1}$$

if the conditions

$$|g^{\cdot}(t)| \leqslant M,\ k(t) = \beta_1(t)\ (0 < \alpha < 1)$$
$$2k(t)g(t) + g^{\cdot}(t)(A, B, C) \geqslant \beta_1(t)\ (\alpha = 1)$$

are satisfied.

Under these conditions, using Theorem 2.4 we obtain that the equilibrium position is uniformly asymptotically stable with respect to $\theta^{\cdot}, \varphi^{\cdot}, \psi^{\cdot}, \theta, \varphi$.

Example 3.3. Consider the motion of a dynamically symmetric satellite whose centre of mass remains at one of the libration points of the restricted circular three-body problem /13/. As in /13 we denote by O_1xyz the system of coordinates rotating with angular velocity Ω about the z axis. The x axis passes through the attracting centres of M_1 and M_2 of mass m_1 and m_2. Point O_1 coincides with the centre of mass m_1 and m_2, and x_1 and x_2 are coordinates of M_1 and M_2; $x, y, z = 0$ are the coordinates of the centre of mass of the satellite $\mu_i = fm_i, r_i^2 = (x - x_i)^2 + y^2\ (i = 1, 2)$ and $A = B$ are its principal central moments of inertia, and θ, φ, ψ are Euler's angles, introduced in the usual way.

Ignoring the cyclic coordinate, we determine the Routh function

$$R = R_2 + R_1 - W$$
$$2R = A(\theta^{\cdot 2} + \psi^{\cdot 2}\sin^2\theta) + 2A\Omega\psi^{\cdot}\sin^2\theta + 2c\psi^{\cdot}\cos\theta + A\Omega^2\sin^2\theta + 2c\Omega\cos\theta - 3(C - A)\sin^2\theta\sum_{i=1}^{2}\mu_i((x - x_i)\sin\psi - y\cos\psi)^2/r_i^5)$$

Since $\partial W/\partial\theta = \partial W/\partial\psi = 0$ when $\theta = 0$, the motions of the satellite are steady /13/

$$\theta^{\cdot} = \theta = 0,\ \varphi^{\cdot} = \text{const} \tag{3.5}$$

where the axis of symmetry of the satellite is perpendicular to the orbital plane.

The function R_2 is positive definite only with respect to θ, but $R_2 \to +\infty$ when $|\sin\theta| \geqslant \delta > 0$ and $|\psi^{\cdot}| \to +\infty$. The function $W - W_0$ is positive definite only with respect to θ only in the case of a point of rectilinear libration, if

$$\begin{aligned} &c\Omega - A\Omega^2 > 0\quad (C > A) \\ &c\Omega - A\Omega^2 + 3\ (C - A)(\mu_1/r_1^3 + \mu_2/r_2^3) > 0\quad (C < A) \end{aligned} \tag{3.6}$$

and in the case of a point of triangular libration, if

$$\begin{aligned} &c\Omega - A\Omega^2 + (C - A)\,y^2(2\mu_1 + 2\mu_2 + d)/r^5 > 0\quad (C < A) \\ &c\Omega - |A\Omega^2 - (C - A)\,y^2(2\mu_1 + 2\mu_2 - d)/r^5| > 0\quad (C > A) \\ &(d = ((\mu_1 + \mu_2)^2 + 3(\mu_1 - \mu_2)^2)^{1/2}) \end{aligned} \tag{3.7}$$

The equations of motion indicate that under conditions (3.6) and (3.7) there are no motions in the region $\{0 < \theta < \pi\}$ along which $\theta = \text{const}$ or $\psi = \text{const}$.

Using Theorem 3.4 and Remarks 3.1, we come to the following conclusion. If the satellite is subjected to dissipative forces whose moments are such that

$$Q_\varphi = 0,\quad Q_\theta\theta^{\cdot} + Q_\psi\psi^{\cdot} \leqslant -\beta_1(t)(\theta^{\cdot})^\alpha\quad (\leqslant -\beta_1(t)\sin^2\theta(\psi^{\cdot})^\alpha)$$

the set of steady motions (3.5) is asymptotically stable with respect to θ relative to the perturbed motions of the satellite with initial conditions satisfying (3.6) or (3.7).

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THE USE OF LYAPUNOV'S SECOND METHOD TO ESTIMATE REGIONS OF STABILITY AND ATTRACTION*

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A definition of the stability region is given by extending the properties of Lyapunov's definition of sets of sizable measure. Constructive theorems on estimates of regions of stability and attraction are obtained by using certain developments of Lyapunov's second method for a wide class of autonomous and non-autonomous systems that satisfy both the Lipschitz and discontinuous conditions. The usual requirements imposed on the functions used in investigations of the stability region are somewhat reduced. For example, the requirement that the functions and their derivatives should have fixed sign are omitted.

1. Consider the equations of perturbed motion of the form

$$x^{\cdot} = f(x, t), \quad x \in R^n \qquad (1.1)-(1.4)$$

By system (1.1) we mean an autonomous system, whose right side is $f(x)$, whose vector function $f(x)$ is such that the solution of the Cauchy problem in the region considered exists, is unique, and is continuous with respect to the initial conditions, excluding any arbitrarily small neighbourhood of singular points. For system (1.2) $f = f(x) \in C(R^n)$ and by Peano's theorem the integral curves can be continued to the boundary of any compact set, possibly in a non-unique way. In system (1.3) the single-valued vector function $f = f(x)$ is piecewise continuous. Among systems (1.3) with discontinuous single-valued right sides only those are considered for which each integral curve may be uniquely continued in the neighbourhood of any surface of discontinuity, and the number of such surfaces is finite. The vector function $f = f(x, t)$ in system (1.4) is such that the solutions retain the properties of the solutions of system (1.1) mentioned above.

The basic concepts and notation correspond to those used in /1/. In addition we shall introduce the upper right Dini derivative /2, 3/ denoted by D^+V; the connected subset F of the semiaxis $[t_0, \infty)$ such that $F = [t_0, T] \vee [t_0, \infty)$ $(T = \text{const})$ (when investigating the properties of attraction $F = [t_0, \infty))$, $F_d^V = \{x \mid V(x) = d\}$; $H_{c(t)}^V = \{x \mid x = y(t, t_0, x_0) \wedge x_0 \in H_{c(t_0)}^V = H_{c_0}^V = \{x \mid V(x) \leqslant c_0\}\}$, $c_0 = \text{const}$, and the integral curve $y(t, t_0, x_0)$ of the system considered under initial conditions x_0, t_0.

Let us assume that for the Lyapunov function $V \in C^1$ the following conditions are satisfied:

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